

On the binary codes with parameters of doubly-shortened 1-perfect codes

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Abstract

We show that any binary $(n = 2^m - 3, 2^{n-m}, 3)$ code C_1 is a part of an equitable partition (perfect coloring) $\{C_1, C_2, C_3, C_4\}$ of the n -cube with the parameters $((0, 1, n-1, 0)(1, 0, n-1, 0)(1, 1, n-4, 2)(0, 0, n-1, 1))$. Now the possibility to lengthen the code C_1 to a 1-perfect code of length $n+2$ is equivalent to the possibility to split the part C_4 into two distance-3 codes or, equivalently, to the biparticity of the graph of distances 1 and 2 of C_4 . In any case, C_1 is uniquely embeddable in a twofold 1-perfect code of length $n+2$ with some structural restrictions, where by a twofold 1-perfect code we mean that any vertex of the space is within radius 1 from exactly two codewords.

The hypercube $H^n = (V(H^n), E(H^n))$ of dimension n is the graph whose vertices are the all binary n -words, two words being adjacent if and only if they differ in exactly one position.

$d(\cdot, \cdot)$ – the Hamming distance, i.e., the natural graph distance in H^n .

$\bar{0} = 0 \dots 0$ (the all-zero word), $\bar{1} = 1 \dots 1$ (the all-one word).

A binary code C of length n and code (or minimal) distance d , or $(n, |C|, d)$ code, is a subset of $V(H^n)$ such that $d(\bar{x}, \bar{y}) \geq d$ for any different \bar{x} and \bar{y} from C .

A partition $\{C_1, \dots, C_r\}$ of $V(H^n)$ into r nonempty parts is said to be *equitable* with parameters $(S_{ij})_{i,j=1}^r$ if for every $i, j \in \{1, \dots, r\}$ every vertex \bar{x} from C_i has exactly S_{ij} neighbors from C_j (the corresponding r -valued function on $V(H^n)$ is known as a *perfect coloring*).

A binary code $C \subset V(H^n)$ is said to be *1-perfect* if every vertex $\bar{x} \in V(H^n)$ is at the distance 0 or 1 from exactly one codeword. Equivalently, $\{C, V(H^n) \setminus C\}$ is an equitable partition with parameters $((0, n)(1, n-1))$. Equivalently, C is a $(2^m - 1, 2^{2^m - m - 1}, 3)$ code, $n = 2^m - 1$.

We will say that a multiset $B \subset V(H^n)$ is a *twofold 1-perfect code* if every vertex $\bar{x} \in V(H^n)$ is at the distance 0 or 1 from exactly two codewords of B . We will say that a multiset $B \subset V(H^n)$ is *splittable* if it can be represented as the (multiset) union of two distance-3 codes; otherwise B is *unsplittable*. The existence of unsplittable twofold 1-perfect codes was proved in [6].

We say that a code C' is obtained by *shortening* from a code $C \subset V(H^n)$ if $C' = \{\bar{x} \in V(H^{n-1}) \mid \bar{x}\bar{0} \in C\}$. Respectively, C'' is *doubly-shortened* from C if

$C'' = \{\bar{x} \in V(H^{n-2}) \mid \bar{x}00 \in C\}$. (Here and elsewhere, for $\bar{x} = x_1x_2\dots x_n$, by $\bar{x}0$ we mean the concatenation of \bar{x} with 0, i.e., the word $x_1x_2\dots x_n0$; similarly we define $\bar{x}1$, $\bar{x}00$, $\bar{x}01$, \dots ; we also expand this notation for sets of words, e.g., $C0 = \{\bar{x}0 \mid \bar{x} \in C\}$.)

It is known [1] that shortened and doubly-shortened (and even triply-shortened) 1-perfect codes have the maximal cardinality among all the codes of the same length and code distance 3. The question [4] is: can every code with such parameters $((2^m - 2, 2^{2^m-m-2}, 3)$ or $(2^m - 3, 2^{2^m-m-3}, 3)$) be represented as a shortened or doubly-shortened 1-perfect code?

For $(2^m - 2, 2^{2^m-m-2}, 3)$ codes the question is solved [2]. In fact, such a code C_1 generates an equitable partition $\{C_1, C_2, C_3\}$ with parameters $((0, n, 0)(1, n - 2, 1)(0, n, 0))$. Then, the code

$$C = C_10 \cup C_31$$

is 1-perfect.

In this paper we prove that a $(2^m - 3, 2^{2^m-m-3}, 3)$ code C_1 generates an equitable partition $\{C_1, C_2, C_3, C_4\}$ with parameters $((0, 1, n - 1, 0)(1, 0, n - 1, 0)(1, 1, n - 4, 2)(0, 0, n - 1, 1))$. If the code C_4 is splittable into two distance-3 codes C' and C'' , then the code

$$C = C_100 \cup C_211 \cup C'01 \cup C''10$$

is 1-perfect. However, the problem of splittability of C_4 remains open. So, the problem of embedding C_100 in a 1-perfect code is unsolved; although, C_100 is proved to be embedded in twofold 1-perfect codes

$$2 \times C_100 \cup 2 \times C_211 \cup C_401 \cup C_410$$

and

$$C_100 \cup C_200 \cup C_111 \cup C_211 \cup C_401 \cup C_410$$

(Theorems 2 and 3), whose splittability is equivalent to the splittability of C_4 .

1 Notation and basic facts

Let C_1 be a binary code of length $n = 2^m - 3$, cardinality 2^{n-m} , and minimal distance 3.

Denote

$$C_2 = C_1 + \bar{1} = \{\bar{x} \mid \bar{x} + \bar{1} \in C_1\}, \quad (1)$$

$$C_3 = \{\bar{x} \mid d(\bar{x}, C_1) = 1\} \setminus C_2, \quad (2)$$

$$C_4 = V(H^n) \setminus (C_1 \cup C_2 \cup C_3); \quad (3)$$

$$A_l^j(\bar{x}) = |\{\bar{y} \in C_j \mid d(\bar{x}, \bar{y}) = l\}|, \quad j \in \{1, 2, 3, 4\}, \bar{x} \in V(H^n)$$

(the tuple $(A_0^i(\bar{x}), A_1^i(\bar{x}), \dots, A_n^i(\bar{x}))$ is known as the weight distribution of C_i with respect to \bar{x}),

$$\bar{A}_l^{ij} = \frac{1}{|C_i|} \sum_{\bar{x} \in C_i} A_l^j(\bar{x}), \quad i, j \in \{1, 2, 3, 4\}, l \in \{0, \dots, n\}$$

(the tuple $(\bar{A}_0^{ii}, \bar{A}_1^{ii}, \dots, \bar{A}_n^{ii})$ is known as the inner distance distribution of C_i).

Best and Brouwer [1] showed that $(2^m - 3, 2^{n-m}, 3)$ codes are optimal, i.e. any $(2^m - 3, M, 3)$ code satisfies $M \leq 2^{n-m}$. Moreover,

Lemma 1 [1]. *The inner distance distribution $(\bar{A}_l^{11})_{l=0}^n$ does not depend on the choice of the $(2^m - 3, 2^{n-m}, 3)$ code C_1 .*

We will also need the following fact:

Lemma 2. *Any 1-perfect or twofold 1-perfect code C is antipodal; i.e., in multiset terms, for any $\bar{x} \in V(H^n)$ the C -multiplicities of \bar{x} and $\bar{x} + \bar{1}$ coincide.*

In the case of 1-perfect codes this is well-known fact, which follows from the results [8, 12]. For twofold 1-perfect codes, the fact has a similar proof. Alternatively, Lemma 2 follows from the fact that the multiplicity function of the considered code is, up to an additive constant, an eigenfunction of H^n with the eigenvalue -1 and the corresponding eigenspace has a simple basis from antipodal functions.

2 An element of equitable partition

Proposition 1. *If C_1 is a doubly-shortened 1-perfect code of length n , then $\bar{A}_n^{11} = \bar{A}_1^{24} = \bar{A}_1^{42} = 0$, $\bar{A}_{n-1}^{11} = \bar{A}_1^{44} = 1$, and $A_1^4(\bar{x}) = \bar{A}_1^{34} = 2$ for any $\bar{x} \in C_3$.*

Proof: Let $C = C_1 \times \{00\} \cup C' \times \{01\} \cup C'' \times \{01\} \cup C''' \times \{11\}$ be a 1-perfect code. If $\bar{x} \in C_1$ (i.e. $\bar{x}00 \in C$), then $\bar{x}00 + \bar{1} \in C$, i.e., $\bar{x} + \bar{1} \in C'''$; so, $C_2 = C'''$ and $\bar{A}_n^{11} = 0$.

If a vertex \bar{y} is at distance at least 2 from C_1 , then, by the definition of a 1-perfect code, the vertex $\bar{y}00$ is at distance 1 from an element of C , which is either $\bar{y}01$ or $\bar{y}10$. So, $\bar{y} \in C' \cup C''$. Vice versa, any $\bar{y} \in C' \cup C''$ is at distance at least 2 from C_1 , because the minimal distance of C is 3. So, $C_4 = C' \cup C''$.

Because of the minimal distance of C , the sets $C_2 = C'''$ and $C_4 = C' \cup C''$ are at distance more than 1 from each other. This means $\bar{A}_1^{24} = \bar{A}_1^{42} = 0$.

We state that for any \bar{x} from C_1 there is exactly one vertex of C_2 at the distance 1 from \bar{x} . Indeed, the vertex $\bar{x}11$ from $V(H^{n+2})$ is at the distance 1 from exactly one codeword of C , which can be only of type $\bar{y}11$, where $\bar{y} \in C_2$ and $d(\bar{x}, \bar{y}) = 1$. Then the vertex $\bar{y} + \bar{1}$ is the only C_1 -vertex at the distance $n - 1$ from \bar{x} ; so, $\bar{A}_{n-1}^{11} = 1$. The remaining part of the proposition is proved by similar arguments. \triangle

We will first prove that

Lemma 3. *All the numbers \bar{A}_l^{ij} ($i, j \in \{1, 2, 3, 4\}, l \in \{0, \dots, n\}$) do not depend on the choice of the $(2^m - 3, 2^{n-m}, 3)$ code C_1 .*

Proof: Once we have proved that \overline{A}_l^{ij} does not depend on the choice C_1 , we know that it is the same as if C_1 would be a double-shortened 1-perfect (for example, Hamming) code. Moreover if it is equal to the minimal or maximal possible value of $A_l^j(\bar{x})$, $\bar{x} \in C_i$, then $A_l^j(\bar{x}) = \overline{A}_l^{ij}$ for any $\bar{x} \in C_i$.

In particular, for any $\bar{x} \in C_1$

$$A_n^1(\bar{x}) = 0 \quad \text{and} \quad A_{n-1}^1(\bar{x}) = 1.$$

This means that the sets C_1 and C_2 are disjoint and

$$\text{any vertex from } C_2 \text{ has exactly one neighbor from } C_1, \text{ and vice versa} \quad (4)$$

(the fact (4) will be used later). So, $\{C_1, C_2, C_3, C_4\}$ is a partition of $V(H^n)$, and we can derive relations between the cardinalities of C_i :

$$|C_2| = |C_1|, \quad |C_3| = (n-1)|C_1|, \quad |C_4| = |V(H^n)| - |C_1| - |C_2| - |C_3| = 2|C_1|.$$

Now we claim the following:

$$\overline{A}_l^{i2} = \overline{A}_{n-l}^{i1} \quad (5)$$

$$\overline{A}_l^{i3} = (n-l+1) \cdot \overline{A}_{l-l}^{i1} + (l+1) \cdot \overline{A}_{l+l}^{i1} - \overline{A}_l^{i3} \quad (6)$$

$$\overline{A}_l^{i4} = \binom{n}{l} - \overline{A}_l^{i1} - \overline{A}_l^{i2} - \overline{A}_l^{i3} \quad (7)$$

$$|C_i| \cdot \overline{A}_l^{ij} = |C_j| \cdot \overline{A}_l^{ji} \quad (8)$$

Indeed, (5) follows from $A_l^2(\bar{x}) = A_{n-l}^1(\bar{x})$, which is straightforward from the definition of C_2 ; (6) follows from $A_l^3(\bar{x}) = (n-l+1) \cdot A_{l-l}^1(\bar{x}) + (l+1) \cdot A_{l+l}^1(\bar{x}) - A_l^3(\bar{x})$, which is straightforward from the definition of C_3 and (4); (7) follows from

$$A_l^4(\bar{x}) + A_l^1(\bar{x}) + A_l^2(\bar{x}) + A_l^3(\bar{x}) = \binom{n}{l}, \quad (9)$$

which is from the fact that $\{C_1, C_1, C_1, C_1\}$ is a partition of $V(H^n)$; the right and left part of (7) are just different ways to calculate the cardinality of $\{(\bar{x}, \bar{y}) \mid \bar{x} \in C_i, \bar{y} \in C_j, d(\bar{x}, \bar{y}) = l\}$.

Starting from \overline{A}_l^{11} , we can calculate \overline{A}_l^{1j} by (5-7), the values of \overline{A}_l^{j1} by (8), the values of \overline{A}_l^{ji} by (5-7); so, Lemma 3 is proved. \triangle

Theorem 1. *The partition $\{C_1, C_2, C_3, C_4\}$ of $V(H^n)$ is equitable with parameters*

$$(S_{ij})_{i,j=1}^4 = \begin{pmatrix} 0 & 1 & n-1 & 0 \\ 1 & 0 & n-1 & 0 \\ 1 & 1 & n-4 & 2 \\ 0 & 0 & n-1 & 1 \end{pmatrix}.$$

Proof: Assume $i, j \in \{1, 2, 3, 4\}$, $\bar{x} \in C_i$. We will show that $A_1^j(\bar{x}) = S_{ij}$.

We have already found (4) that $A_1^j(\bar{x}) = 1$ if $(i, j) \in \{(1, 2), (2, 1)\}$. Since C_1 and C_2 are distance-3 codes, $A_1^j(\bar{x}) = 0$ if $(i, j) \in \{(1, 1), (2, 2)\}$. Then, by the definition of C_4 , we have $A_1^j(\bar{x}) = 0$ if $(i, j) \in \{(1, 4), (4, 1)\}$.

By (9) we get $A_1^j(\bar{x}) = n - 0 - 1 - 0 = n - 1$ if $(i, j) = (1, 3)$.

Since $\overline{A}_1^{24} = \overline{A}_1^{42} = 0$, we also have $A_1^j(\bar{x}) = 0$ if $(i, j) \in \{(2, 4), (4, 2)\}$.

By (9), $A_1^j(\bar{x}) = n - 1 - 0 - 0 = n - 1$ if $(i, j) = (2, 3)$.

Let us check that $A_1^j(\bar{x}) = 1$ if $(i, j) = (4, 4)$. Since $\overline{A}_1^{44} = 1$ (Proposition 1), it is enough to prove that $A_1^j(\bar{x})$ is odd. Indeed, as follows from the arguments above, the neighborhood of \bar{x} consists of only C_3 - and C_4 -vertices. Every such C_3 -vertex is adjacent with exactly one C_1 -vertex, which is at distance 2 from \bar{x} . While every such C_1 -vertex is adjacent with exactly two vertices from the neighborhood of \bar{x} . So, this neighborhood contains an even number of vertices from C_3 and, consequently, an odd, from C_4 .

Automatically, we get $A_1^j(\bar{x}) = n - 0 - 0 - 1 = n - 1$ if $(i, j) = (4, 3)$.

Let us show that $A_1^j(\bar{x}) = 2$ if $(i, j) = (3, 4)$. We will calculate the number T of triples $\{\bar{a}, \bar{b}, \bar{c}\}$ such that $\bar{b} \in C_3$ is adjacent to both $\bar{a}, \bar{c} \in C_4$. At first, we observe that $T = |C_4| \overline{A}_2^{44}$ is independent on the choice of C_1 . At second, it can be calculated as

$$\sum_{\bar{b} \in C_3} \frac{A_1^4(\bar{b})(A_1^4(\bar{b}) - 1)}{2} = \frac{1}{2} \sum_{\bar{b} \in C_3} (A_1^4(\bar{b}))^2 - \frac{1}{2} |C_3| \overline{A}_1^{34};$$

so, by the Cauchy–Bunyakovsky inequality,

$$T \geq \frac{1}{2|C_3|} \left(\sum_{\bar{b} \in C_3} A_1^4(\bar{b}) \right)^2 - \frac{|C_3|}{2} \overline{A}_1^{34} = \frac{|C_3|}{2} \overline{A}_1^{34} (\overline{A}_1^{34} - 1),$$

where the equality holds if and only if all $A_1^4(\bar{b})$, $\bar{b} \in C_3$, are equal to the same value (i.e., to $\overline{A}_1^{34} = 2$). But the last is true when C_1 is a doubly-shortened 1-perfect code (Proposition 1); consequently, it is true for any $(2^m - 3, 2^{n-m}, 3)$ code.

Finally, if $(i, j) = (3, 3)$, then $A_1^j(\bar{x}) = n - 1 - 1 - 2 = n - 4$. \triangle

Remark 1. 1) If we unify the two parts C_1 and C_2 , say $C_{12} = C_1 \cup C_2$, then we will obtain an equitable partition $\{C_{12}, C_3, C_4\}$ with parameters

$$\begin{pmatrix} 1 & n-1 & 0 \\ 2 & n-4 & 2 \\ 0 & n-1 & 1 \end{pmatrix}. \quad (10)$$

We see that the parameter matrix is symmetrical with respect to interchanging of the parts C_{12} and C_4 . But C_{12} is known to be splittable, while the splittability of C_4 is questionable. When C_1 is a doubly-shortened 1-perfect code, we know that both C_{12} and C_4 are splittable. Moreover, one can construct an equitable partition with parameters (10) whose first and third parts are unsplittable. The problem is if there exists such a partition with exactly one of C_{12} and C_4 being splittable.

2) If C_4 is splittable, then after splitting it, from the partition $\{C_1, C_2, C_3, C_4\}$ we obtain an equitable partition with parameters

$$\begin{pmatrix} 0 & 1 & n-1 & 0 & 0 \\ 1 & 0 & n-1 & 0 & 0 \\ 1 & 1 & n-4 & 1 & 1 \\ 0 & 0 & n-1 & 0 & 1 \\ 0 & 0 & n-1 & 1 & 0 \end{pmatrix},$$

which also have some obvious symmetries.

Remark 2. An equitable partition with $\{C_{12}, C_3, C_4\}$ of H^n with parameters (10) generates an equitable partition $\{G_1, G_2, G_3, G_4\}$ of $H^{n'}$, $n' = n + 1$ with parameters

$$\begin{pmatrix} 0 & n' & 0 & 0 \\ 2 & 0 & n'-2 & 0 \\ 0 & n'-2 & 0 & 2 \\ 0 & 0 & n' & 0 \end{pmatrix}$$

as follows:

$$\begin{aligned} G_1 &= \{\bar{x}\alpha \mid \bar{x} = x_1x_2\dots x_n \in C_{12}, \alpha = x_1 + \dots + x_n \bmod 2\}, \\ G_4 &= \{\bar{x}\beta \mid \bar{x} = x_1x_2\dots x_n \in C_4, \beta = x_1 + \dots + x_n + 1 \bmod 2\}, \\ G_2 &= \{\bar{y} \in V(H^{n'}) \mid d(\bar{y}, C_1) = 1\}, \\ G_3 &= \{\bar{y} \in V(H^{n'}) \mid d(\bar{y}, C_4) = 1\}. \end{aligned}$$

This partition can be viewed as an “extended” version of the partition $\{C_{12}, C_3, C_4\}$; the spittability of C_{12} or C_4 is equivalent to the spittability of G_1 or G_4 respectively. But the distance 1 between vertices of, say, C_4 corresponds to the distance 2 between the corresponding vertices of G_4 ; and the graph of distances 1 and 2 of C_4 corresponds to the graph of distances 2 of G_4 , which emphasize the “equal status” of the all edges of the graph. Of cause if G_1 and/or G_4 are splittable, then splitting gives an equitable partition of $H^{n'}$ into 6 / 5 parts with corresponding parameters. If both G_1 and G_4 are splittable (say, into G'_1, G''_1 and G'_4, G''_4 respectively), then the equitable partition $\{G_1, G_2, G_3, G_4\}$ (defined in some other terms) is also known as a *code-generating factorization* of $H^{n'}$ [14]. Indeed, the code $G'_10 \cup G''_11 \cup G'_40 \cup G''_41$ is 1-perfect.

3 Embedding in twofold 1-perfect codes

Theorem 2. Let C_1 be a $(n = 2^m - 3, 2^{2^m - m - 3}, 3)$ code. Then the set $C_100 = \{\bar{x}00 \mid \bar{x} \in C_1\}$ is a subset of a unique twofold 1-perfect code B with the following properties:

- a) the multiplicity of any codeword of C_100 is 2;
- b) any codeword \bar{x} with the last two symbols 01 or 10 satisfies $\bar{x} + 0\dots 011 \in C$.

Proof: Existence. Let $B = 2 * C_1 00 \cup 2 * C_2 11 \cup C_4 01 \cup C_4 10$. Obviously, B satisfies a), b), and $C_1 00 \subset B$. The fact that B is a twofold 1-perfect code is straightforward from Theorem 1; we leave the details as an exercise.

Uniqueness. Assume B is a twofold 1-perfect code satisfying a), b), and $C_1 00 \subset B$. Define

$$\begin{aligned} C_2 &= \{\bar{x} \mid \bar{x} 11 \in B\}, \\ C_4 &= \{\bar{x} \mid \bar{x} 01 \in B\}, \\ C_3 &= V(H^n) \setminus (C_1 \cup C_2 \cup C_4) \end{aligned}$$

From the antipodality of B , we have $C_2 = C_1$. As follows from the definition of twofold 1-perfect codes, any codeword of multiplicity 2 cannot be at distance 1 or 2 from any other codeword. Consequently, 1) the distance between C_1 and C_4 , as well as between C_2 and C_4 , cannot be less than 2; 2) the multiplicity of the words of form $\bar{x} 01$ in B is less than 2.

Now we see that, by numerical reasons, C_4 consists of the all vertices at the distance more than 1 from C_1 . Thus, C_2 , C_3 , and C_4 satisfy (1)-(3), and B is unique. \triangle

By similar arguments, the following is also true:

Theorem 3. *Let C_1 be a $(n = 2^m - 3, 2^{2^m - m - 3}, 3)$ code. Then the set $C_1 00$ is a subset of a unique twofold 1-perfect code D whose all codewords \bar{x} satisfy $\bar{x} + 0 \dots 011 \in C$.*

4 Embedding in 1-perfect codes

Theorem 4. *Let C_1 be a $(n = 2^m - 3, 2^{2^m - m - 3}, 3)$ code. The following four statements are mutually equivalent:*

- a) *the set $C_1 00$ is a subset of a 1-perfect code C ;*
- b) *the set C_4 defined in (3) is splittable;*
- c) *the twofold 1-perfect B from Theorem 2 is splittable;*
- d) *the twofold 1-perfect D from Theorem 3 is splittable.*

Proof: Clearly, each of c) and d) implies a).

Since B , as well as D , includes $C_4 01$, each of c) and d) implies b).

Conversely, assume b) holds and $C_4 = C' \cup C''$ where C' and C'' are distance-3 codes. Then

$$\begin{aligned} B &= (C_1 00 \cup C_4 11 \cup C' 01 \cup C'' 10) \cup (C_1 00 \cup C_4 11 \cup C' 10 \cup C'' 01), \\ D &= (C_1 00 \cup C_4 11 \cup C' 01 \cup C'' 10) \cup (C_4 00 \cup C_1 11 \cup C' 10 \cup C'' 01), \end{aligned}$$

and c), d) hold.

Assume a) is true. Define $C' = \{\bar{x} \mid \bar{x} 01 \in C\}$ and $C'' = \{\bar{x} \mid \bar{x} 10 \in C\}$. Because of the code distance 3 of C , we see that C' and C'' are disjoint and at the distance at least 2 from C_1 . So, since $|C'| + |C''| = |C_4|$, we get $C_4 = C' \cup C''$, and b) holds. \triangle

Remark 3. The splittability of any of the sets C_4 , B , D is equivalent to the biparticity of its graph of distances 1 and 2 (two codewords \bar{x} and \bar{y} are adjacent if and only if $d(\bar{x}, \bar{y}) \in \{1, 2\}$). In this graph for D , the vertices of types $\bar{x}00$ and $\bar{x}11$ are not connected with the vertices of types $\bar{x}01$ and $\bar{x}10$, and the subgraph generated by the former vertices is bipartite, while the biparticity of the remaining subgraph is questionable. In B , the codewords of types $\bar{x}00$ and $\bar{x}11$ have the multiplicity 2, and they are isolated in the graph of distances 1 and 2.

Remark 4. If ν is the number of connected components in the graph of distances 1 and 2 of C_4 , then the number of different 1-perfect codes including C_100 is 2^ν . As follows from the tight lower bound on the size of the difference between two 1-perfect codes [13, 3], the cardinality of a connected component is not less than $2^{\frac{n-1}{2}}$, and so $\nu \geq \frac{2^{\frac{n-3}{2}}}{n+1}$. If C_1 is linear, then ν achieves this bound.

5 Unsplittable twofold STS

If we consider a 1-perfect code containing $\bar{0}$, then all the weight-3 codewords compose a design known as a Steiner triple system, or STS. The characteristic property of an STS is that every weight-2 word is at distance 1 from exactly one word of the STS. (Strictly speaking, an STS is defined as a pair (V, B) , where V is some set and B is a collection of 3-subsets of V , named blocks, such that every 2-subset of V is included in exactly one block.)

If we consider a twofold 1-perfect code C such that the multiplicity of $\bar{0}$ is 2, then all the weight-3 codewords compose a design, which can be called a twofold STS. If C comes from Theorem 2, then the corresponding STS satisfies

- a) any codeword of type $\bar{x}00$ or $\bar{x}11$ has the multiplicity 2;
- b) for any \bar{x} of the corresponding length, $\bar{x}01$ and $\bar{x}10$ are codewords or not simultaneously.

For the length 15, there exists a twofold STS meeting a) and b) that cannot be split into two STS. This fact has not direct connection with the problem considered in this paper: on one hand, it is not proved that there exists a twofold 1-perfect code that include this STS (e.g., for the length 15, there exist STSs that are not embeddable in a 1-perfect code [10]); on the other hand, the splittability of the all twofold STS included in a twofold 1-perfect code would not mean the splittability of the twofold 1-perfect itself. Nevertheless, the existence of such an object seems to be interesting. The following is the list of the words of the mentioned example (the unsplittability follows from the existence of a 5-cycle in the distance-2 graph):

```
0000000 00000 1 11 × 2 ,
0000000 11000 0 01 ,      0000000 11000 0 10 ,
0000000 01100 0 01 ,      0000000 01100 0 10 ,
0000000 00110 0 01 ,      0000000 00110 0 10 ,
0000000 00011 0 01 ,      0000000 00011 0 10 ,
0000000 10001 0 01 ,      0000000 10001 0 10 ,
1100000 00000 0 01 ,      1100000 00000 0 10 ,
0110000 00000 0 01 ,      0110000 00000 0 10 ,
0011000 00000 0 01 ,      0011000 00000 0 10 ,
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0001100 00000 0 01 ,	0001100 00000 0 10 ,
0000110 00000 0 01 ,	0000110 00000 0 10 ,
0000011 00000 0 01 ,	0000011 00000 0 10 ,
1000001 00000 0 01 ,	1000001 00000 0 10 ,
0000000 01010 1 00 $\times 2$,	
0100010 00000 1 00 $\times 2$,	
0010100 00000 1 00 $\times 2$,	
0001000 00100 1 00 $\times 2$,	
1000000 10000 1 00 $\times 2$,	0000001 00001 1 00 $\times 2$,
0100000 10100 0 00 $\times 2$,	0000010 00101 0 00 $\times 2$,
0010000 10010 0 00 $\times 2$,	0000100 01001 0 00 $\times 2$,
1010000 00001 0 00 $\times 2$,	0000101 10000 0 00 $\times 2$,
0101000 00001 0 00 $\times 2$,	0001010 10000 0 00 $\times 2$,
1000010 00010 0 00 $\times 2$,	0100001 01000 0 00 $\times 2$,
1001000 01000 0 00 $\times 2$,	0001001 00010 0 00 $\times 2$,
0100100 00010 0 00 $\times 2$,	0010010 01000 0 00 $\times 2$,
1000100 00100 0 00 $\times 2$,	0010001 00100 0 00 $\times 2$.

6 MDS codes and double-MDS-codes

Let $Q^m = (V(Q^m), E(Q^m))$ denotes the graph whose vertex set is the set $\{0, 1, 2, 3\}^m$ of quaternary n -words, two words being adjacent if and only if they differ in exactly one position. By a 4-*clique* we mean a set of four words of $V(Q^m)$ differing in exactly one position.

A subset M of $V(Q^m)$ is called an MDS code (with distance 2) if every 4-*clique* contains exactly one word of M . Equivalently, M is a distance 2 code of cardinality 4^{m-1} . Equivalently, $\{M, V(Q^m) \setminus M\}$ is an equitable partition of Q^m with parameter matrix $((0, 3m)(m, 2m))$.

We call a subset M of $V(Q^m)$ a double-MDS-code if every 4-*clique* contains exactly two word of M . Equivalently, $\{M, V(Q^m) \setminus M\}$ is an equitable partition of Q^m with parameter matrix $((m, 2m)(2m, m))$.

A double-MDS-code is *splittable* if it is the union of two (disjoint) MDS codes.

Denote $P_0 = \{0000, 1111\}$, $P_1 = \{0011, 1100\}$, $P_2 = \{0101, 1010\}$, $P_3 = \{0110, 1001\} \subset V(H^4)$ and $P'_0 = \{000, 111\}$, $P'_1 = \{011, 100\}$, $P'_2 = \{101, 010\}$, $P'_3 = \{110, 001\} \subset V(H^3)$. Let $C \in V(H^{m-1})$ be a 1-perfect binary code; denote $C^* = \{000c_1 000c_2 \dots 000c_{m-1} 000 \mid c_1 c_2 \dots c_{m-1} \in C\}$. For any subset M of $V(Q^m)$ we define the code $S(M) \subset V(H^{4m-1})$ as follows:

$$S(M) = \bigcup_{\mu_1 \dots \mu_m \in M} P_{\mu_1} P_{\mu_2} \dots P_{\mu_{m-1}} P'_{\mu_m} + C^* \quad (11)$$

Here, for two sets of words $P^1 \subset V(H^r)$ and $P^2 \subset V(H^l)$, $P^1 P^2 = \{x_1 \dots x_r y_1 \dots y_l \mid x_1 \dots x_r \in P^1, y_1 \dots y_l \in P^2\}$; and if $r = l$, then $P^1 + P^2 = \{z_1 \dots z_r \mid z_i = x_i + y_i \bmod 2, x_1 \dots x_r \in P^1, y_1 \dots y_r \in P^2\}$.

Proposition 2. 1) If the code distance of M is not less than 2, then the code distance of $S(M)$ is at least 3; if M is an MDS code, then $S(M)$ is a 1-perfect code. 2) If M is a splittable (unsplittable) double-MDS-code, then $S(M)$ is a splittable (unsplittable) twofold 1-perfect code.

Proof (a sketch): P. 1) is proved in [11], in more general form.

Similarly, if M is a double-MDS-code, then $S(M)$ is a twofold 1-perfect code (it is straightforward to check the definition). If $M = M' \cup M''$ for some MDS codes M' and M'' , then $S(M) = S(M') \cup S(M'')$, where $S(M')$ and $S(M'')$ are 1-perfect codes. Otherwise, the distance-1 graph of M has an odd cycle, and it is easy to find a corresponding cycle of the same length in the graph of distances 1 and 2 of $S(M)$, which implies that $S(M)$ is unsplittable. \triangle

Theorem 5. *Let $m = 2^{k-2}$. Assume there exists an unsplittable double-MDS-code $M_1 \subset V(Q^{m-1})$ such that the double-MDS-code $M_0 = V(Q^{m-1}) \setminus M_1$ is splittable. Then there exist a $(n = 2^k - 3, 2^{2^k-k-3}, 3)$ code C_1 such that C_100 is not a subset of a 1-perfect code.*

Proof: Let $M_0 = M' \cup M''$ where M' and M'' are disjoint MDS codes.

Denote $M = M_00 \cup M_01 \cup M_12 \cup M_13 \subset V(Q^m)$. By the definition, M is a double-MDS-code. Since M_1 is unsplittable, M is unsplittable too. Then, by Proposition 2, the set

$$D = S(M)$$

is an unsplittable twofold 1-perfect code.

Now, consider the set

$$C = S(M'0 \cup M'1).$$

Since the code distance of $M'0 \cup M'1$ is 2, the code distance of C is at least 3, by Proposition 2. Half of the codewords of C have 00 in the last two positions (the others, 11); let C_100 denote the corresponding subcode.

We have: $|C_1| = \frac{1}{8}|C| = 2^{2^k-k-3}$; the code distance of C_1 is 3; $C_100 \subset D$ where D is an unsplittable twofold 1-perfect code whose all codewords \bar{x} satisfy $\bar{x} + 0...011 \in C$. By Theorems 3 and 4, the proof is over. \triangle

Conjecture (V. Potapov). *Any double-MDS-code M in Q^m is splittable if and only if its complement $V(Q^m) \setminus M$ is splittable.*

This is equivalent to the following statement: *Any $4 \times 4 \times \dots \times 4 \times 2$ latin hypercuboid is completable to a $4 \times 4 \times \dots \times 4 \times 4$ latin hypercube.* A $q \times q \times \dots \times q \times p$ latin hypercuboid of order q (if $p = q$, latin hypercube) is a function $f : \{0, \dots, q-1\}^{m-1} \times \{0, \dots, p-1\} \rightarrow \{0, \dots, q-1\}$ such that $f(\bar{x}) \neq f(\bar{y})$ for any \bar{x} and \bar{y} differing in exactly one position. Examples of non-completable latin cuboids are constructed in [5, 9]

Another equivalent formulation: *Let K_4^m be the direct product of m copies of the complete graph on 4 vertices. If $V(K_4^m)$ is partitioned into two subsets that generate subgraphs of degree m , then these subgraphs are bipartite or not bipartite simultaneously.*

It seems perspective to use the characterization of the distance-2 MDS codes over the quaternary alphabet (latin hypercubes of order 4) [7] to prove this conjecture. Nevertheless, the analysis of all subcases needs some work, which is not completed at this moment. In any case, it is interesting to find an independent proof.

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